UNIVERSITY OF BOLTON

SCHOOL OF HEALTH & SOCIAL SCIENCES

MATHEMATICS PATHWAY

SEMESTER 1 EXAMINATIONS 2009/10

TOPOLOGY

MODULE NO: MAS3012

Date: Monday 18 January 2010 **Time: 2.00 - 4.15pm**

INSTRUCTIONS TO CANDIDATES:

1. **There are SIX questions.**
2. **Answer FOUR questions.**
3. **All questions carry equal marks.**
4. **Maximum marks for each part/question are shown in brackets.**

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1. (a) Let U and V be subsets of $\boldsymbol{R}^N$.

State what it means for U to be *topologically equivalent* to V.

By defining a suitable mapping show that the rectangle

$$U = \{(x, y) : 0 \le x \le 1,\ \ 0 \le y \le 2\}$$

is topologically equivalent to the rectangle

$$V = \{(x, y) : 0 \le x \le 2,\ \ 0 \le y \le 1\}.$$

(8 marks)

(b) State what it means for a subset of $\boldsymbol{R}^N$ to be *path-connected*.

Show that if U is topologically equivalent to V, and U is path-connected, then V is also path-connected.

(11 marks)

(c) Sketch the region $V = \{(x, y) : 1 \le |x| \le 2,\ \ 0 \le y \le 1\}$ and show that it is not path-connected.

(6 marks)

Please turn the page

2. (a) Let $\sigma : [0,1] \to U$ be a path in $U \subseteq \boldsymbol{R}^N$ from x to y.

State how the *reverse path* $\bar{\sigma}$ is defined.

Verify that the mapping $H : [0,1] \times [0,1] \to U$ given by

$$H(s,t) = \begin{cases} \sigma(2st) & \text{for } 0 \le s \le \frac{1}{2} \\ \sigma(2(1-s)t) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

is a homotopy between the paths $\sigma * \bar{\sigma}$ and the constant path ε_x.

(9 marks)

(b) Suppose $U, V \subseteq \boldsymbol{R}^N$ and let $f : U \to V$ be a homeomorphism.

Let $x \in U$ and consider the fundamental groups $\pi_1(U,x)$ and $\pi_1(V, f(x))$.

Show that the mapping $F : \pi_1(U,x) \to \pi_1(V, f(x))$ given by $F([\sigma]) = [f \circ \sigma]$ is well-defined.

(10 marks)

(c) Using suitable graphs explain why the fundamental group of the projective plane is

$$\pi_1(P) \cong \boldsymbol{Z}_2.$$

(6 marks)

Please turn the page

3. (a) Explain how the torus may be constructed by identifying pairs of edges of a square. Hence write down the symbol for the torus, T.

(5 marks)

(b) By constructing the torus as a simplicial complex and counting faces, edges and vertices find the Euler characteristic $\chi(T)$.

Let $T^n = \underbrace{T\#T\#...\#T}_{n}$

Prove by induction that $\chi(T^n) = 2 - 2n$.

(10 marks)

(c) Suppose that S is a symbol for a surface in which each letter appears twice with *opposite* exponents.

Show that the surface must be of the form

$$T\#T\#...\#T.$$

You may use the following rules for equivalence of symbols:

(I) $...aBCa^{-1}... \sim ...aCBa^{-1}...$

(II) $...aBa^{-1}C... \sim ...CaBa^{-1}...$

(10 marks)

Please turn the page

4. (a) A chain complex is a set of vector spaces over $\boldsymbol{Z}_2$, with linear transformations as follows:

$$\ldots \xrightarrow{\phi_{i+2}} C_{i+1} \xrightarrow{\phi_{i+1}} C_i \xrightarrow{\phi_i} C_{i-1} \xrightarrow{\phi_{i-1}} \ldots$$

State what it means for a pair of i-cycles to be *homologous*.

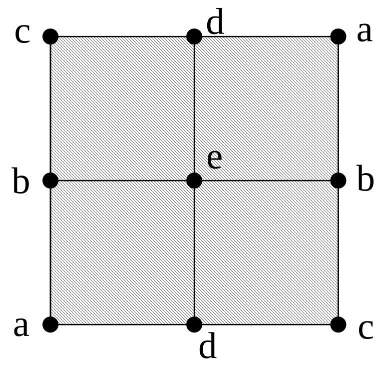
Let $[\alpha]$ denote the homology class of an i-cycle α. State how a binary operation is defined on the set of homology classes of i-cycles to form a group.

Show that this binary operation is well-defined.

Verify the group axioms.

(13 marks)

(b) Consider the projective plane P, formed by identifying pairs of edges as shown in the following diagram:



(i) Describe the 0-cycles and explain why $H_0(P) \cong \boldsymbol{Z}_2$.

(4 marks)

(ii) Give an example of a non-zero 1-cycle that is homologous to 0, and an example of a 1-cycle that is *not* homologous to 0.

State $H_1(P)$.

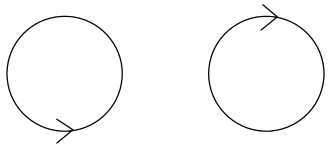
(4 marks)

(iii) Explain why 0 is the only 2-boundary . List the 2-cycles, and hence find $H_2(P)$.

(4 marks)

Please turn the page

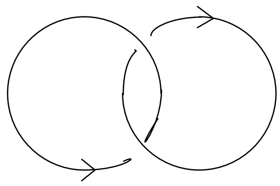
5. (a) Consider the 2-component unlink:



Show that the Conway polynomial of this link is 0.

(6 marks)

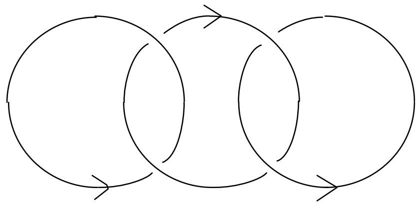
(b) Consider the Hopf link:



Show that the Conway polynomial of this link is $-z$.

(7 marks)

(c) Calculate the Conway polynomial for the following link of three components:



(8 marks)

(d) Explain what it means for a knot K to be *prime*.

(4 marks)

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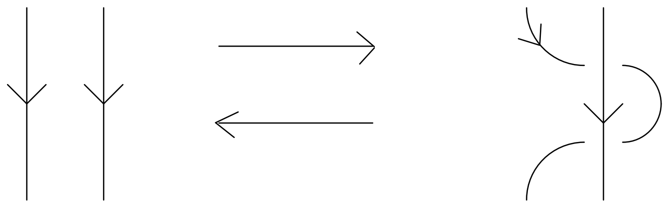
6. (a) Consider a Type II Reidemeister move, as shown in the diagram below:

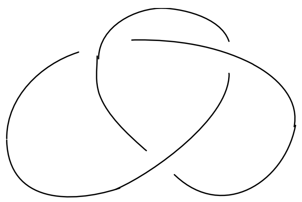


Explain why the writhe of a diagram is unchanged when a Type II move is carried out.

Show that the bracket polynomial is unchanged by a Type II move.

(9 marks)

(b) Calculate the Jones polynomial for the trefoil knot below:



(12 marks)

(c) State the Jones polynomial for the mirror image of the trefoil of (b).

State what this allows us to deduce about the trefoil.

(4 marks)

END OF QUESTIONS