UNIVERSITY OF BOLTON

SCHOOL OF HEALTH & SOCIAL SCIENCES

MATHEMATICS PATHWAY

SEMESTER 1 EXAMINATIONS 2009/10

FURTHER MATHEMATICAL METHODS

MODULE NO: MAS2506

Date: Tuesday 19 January 2010 **Time: 14.00 - 16.15**

INSTRUCTIONS TO CANDIDATES:

1. There are SIX questions.
2. Answer FOUR questions.
3. All questions carry equal marks.
4. Maximum marks for each part/question are shown in brackets.

Health & Social Sciences
Mathematics Pathway
Semester 1 Examinations 2009/10
Further Mathematical Methods
MAS2506

1. (a) Find the general solution of the differential equation

$$3x\frac{dy}{dx} = 4 - x^2$$

(3 marks)

(b) The equation

$$\frac{dv}{dt} = -(av + bt),$$

where a and b are constants, represents the motion of a particle as it moves through a resistive medium.

Solve the equation for v given that $v = u$ when $t = 0$.

(10 marks)

(c) Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 20e^{-2x}$$

given $y = 3$ and $\frac{dy}{dx} = 1$ when $x = 0$.

(12 marks)

2. (a) Explain with the aid of suitable diagrams what is meant by the terms 'odd function' and 'even function'.

Sketch the periodic function defined by:

$$f(t) = \begin{cases} -t & -2 < t < 0 \\ t & 0 < t < 2 \end{cases}$$

and find its general Fourier series.

Expand this series as far as the fifth harmonic.

(15 marks)

(b) Find a half range sine series for the periodic function

$$f(x) = \begin{cases} 3 & 0 < x < \pi/2 \\ 1 & \pi/2 < x < \pi \end{cases}$$

(10 marks)

Please turn the page

3. (a) Use double integration to find the co-ordinates of the centroid of the area bounded by the curve $x = \frac{1}{2}y^2$ and the lines $x = 0$ and $y = 4$.

(10 marks)

(b) With the aid of a suitable diagram reverse the order of integration in the integral

$$\int_0^3 \int_{2y-6}^{\frac{1}{2}(3-y)} x^2 \, dx \, dy$$

and evaluate the resulting integrals.

(15 marks)

4. (a) Using the definition of the Laplace Transform, show that

(i) $L\{a\} = \frac{a}{s}$ (2 marks)

(ii) $L\{e^{at}\} = \frac{1}{s-a}$ (3 marks)

where $a \in \mathbb{R}$.

(b) Solve the simultaneous differential equations

$$4\frac{dx}{dt} - 2\frac{dy}{dt} + 10x - 5y = 0$$

$$\frac{dy}{dt} - 18x + 15y = 0$$

given that $y(0) = 4$ and $x(0) = 2$.

(16 marks)

(c) Using $e^{i\theta} = \cos\theta + i\sin\theta$ or otherwise, show that

(i) $L\{\sin at\} = \frac{a}{s^2 + a^2}$

(ii) $L\{\cos at\} = \frac{s}{s^2 + a^2}$

where $a \in \mathbb{R}$.

(4 marks)

Please turn the page

5. (a) Locate the saddle points on the surface

$$z = x^3 + y^3 - 12x - 12y + 5.$$

Find the value of z at each saddle point.

(10 marks)

(b) A closed rectangular tank is to be made of sheet metal and is to have a volume of 50m^3.

Use partial differentiation to determine the dimensions and the total surface area of the tank so that the area of sheet metal used is a minimum.

(15 marks)

6. (a) Show that if a pair of sequences x_n and y_n have z-transforms $X(z)$ and $Y(z)$ respectively, then the sequence $x_n + y_n$ has z-transform $X(z) + Y(z)$.

(4 marks)

(b) Use z-transforms to solve the difference equation

$$2f(n+2) + f(n+1) - 6f(n) = 2^n$$

given that $f(0) = 0$ and $f(1) = 2$.

(14 marks)

(c) Show that if a sequence x_n, $n \geq 0$, has z-transform $X(z)$, then the sequence x_{n+1} has z-transform $zX(z) - zx_0$.

Hence, or otherwise, show that the sequence 3^{n+1} has z-transform $\dfrac{3z}{z-3}$.

(7 marks)

END OF QUESTIONS

Further Mathematical Methods
Formula Sheet

1. Quadratic Equation

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. Laws of Logarithms

$$\log AB = \log A + \log B$$
$$\log \tfrac{A}{B} = \log A - \log B$$
$$\log A^p = p \log A$$

3. Trigonometry

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1$$

$$\tan^2\theta + 1 \equiv \sec^2\theta$$

$$\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

4. Hyperbolic Functions

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$$

5. Partial Fractions

Type (Denominator containing)	Expression	Form of Partial Fractions
Linear factors	$\dfrac{f(x)}{(x+a)(x+b)(x+c)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+b)}+\dfrac{C}{(x+c)}$
Repeated linear factors	$\dfrac{f(x)}{(x-a)^3}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-a)^3}$
Quadratic factors	$\dfrac{f(x)}{(ax^2+bx+c)(x-d)}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{(x-d)}$
General	$\dfrac{f(x)}{(x^2+a)(x+b)^2(x+c)}$	$\dfrac{Ax+B}{(x^2+a)}+\dfrac{C}{(x+b)}+\dfrac{D}{(x+b)^2}+\dfrac{E}{(x+c)}$

6. Complex Numbers

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

7. Second Order Differential Equations

For the equation $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$ the auxiliary equation is $am^2+bm+c=0$

Roots of auxiliary equation	General solution of the differential equation
2 real roots, m_1 and m_2	$y = Ae^{m_1x}+Be^{m_2x}$
2 equal roots, m	$y=(Ax+B)e^{mx}$
Complex roots, $p \pm iq$	$y=e^{px}(A\cos qx + B\sin qx)$

8. Fourier Series

For a function $f(x)$ of period 2π,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n \sin nx\}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

9. Basic Calculus

If $y = u.v$ then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

If $y = \frac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If y is a function of u and u is a function of x then $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx}$

Integration by parts: $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

10. Laplace Transforms

Definition: $L(f(t)) = \bar{f}(s) = \displaystyle\int_0^\infty e^{-st} f(t)\,dt$

$f(t)$	$L(f) = \bar{f}(s)$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$
e^{kt}	$1/(s-k)$
$\sin \omega t$	$\omega/(s^2+\omega^2)$
$\cos \omega t$	$s/(s^2+\omega^2)$
dx/dt	$sL(x) - x_0$
d^2x/dt^2	$s^2L(x) - sx_0 - \dot{x}_0$
$e^{-at}f(t)$	$\bar{f}(s+a)$
$\int_0^t f(t-x)g(x)dx$	$\bar{f}(s)\bar{g}(s)$
$H(t-a)$	e^{-as}/s
$f(t-a)H(t-a)$	$e^{-as}\bar{f}(s)$

Heavyside Unit Step Function:

Definition: $H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

if $a = 0$

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

1. $L\{H(t-a)\} = \frac{e^{-as}}{s}$ $\qquad s > 0$
2. $L\{H(t)\} = \frac{1}{s}$
3. If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(t-a).H(t-a)\} = e^{-as}\bar{f}(s)$

11. Stationary Points

If $z = f(x,y)$ then

(i) for stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

(ii) $\Delta = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$

if $\Delta > 0$ stationary point is a saddle point.

if $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$ stationary point is a maximum point.

if $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ stationary point is a minimum point.

12. Z Transforms

Geometric Progression: $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$

Definition: $Z(x_n)_{n\geq 0} = X(z) = \sum_{n=0}^{\infty}\left(\frac{x_n}{z^n}\right)$

$\{x_n\}$ $(n \geq 0)$	$z\{x_n\}$	Region of existence
$x_n = \begin{cases} 1 & (n=0) \\ 0 & (n>0) \end{cases}$ (unit pulse sequence)	1	All z
$x_n = 1$ (unit step sequence)	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$x_n = a^n$ (a constant)	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$x_n = n$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$x_n = na^{n-1}$ (a constant)	$\frac{z}{(z-a)^2}$	$\lvert z\rvert > a$
$x_n = e^{-nT}$ (T constant)	$\frac{z}{z-e^{-T}}$	$\lvert z\rvert > e^{-T}$
$x_n = \cos n\omega T$ (ω, T constants)	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$	$\lvert z\rvert > 1$
$x_n = \sin n\omega T$ (ω, T constants)	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$	$\lvert z\rvert > 1$

13. Difference Equations

If $X(z) = Z(x_n)$ then

$$Z(x_{n+1}) = zX(z) - zx_0$$

$$Z(x_{n+2}) = z^2X(z) - z^2x_0 - zx_1$$

CALCULUS

(* the constant of integration has been omitted *)

$\int y\,dx$	y	$\frac{dy}{dx}$
$\left.\begin{matrix}\frac{x^{n+1}}{n+1} & n \neq -1 \\ \ln x & n = -1\end{matrix}\right\}$	x^n	nx^{n-1}
$x \ln\lvert x\rvert - x$	$\ln x$	$\frac{1}{x}$
$\frac{1}{a}e^{ax}$	e^{ax}	ae^{ax}
$\frac{a^x}{\ln a}$	$a^x \quad a > 0$	$a^x \ln a$
$\frac{(ax+b)^{n+1}}{a(n+1)}$	$(ax+b)^n$	$na(ax+b)^{n-1}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$	
$-\frac{1}{a}\cos ax$	$\sin ax$	$a\cos ax$
$\frac{1}{a}\sin ax$	$\cos ax$	$-a\sin ax$
$\frac{1}{a}\ln(\sec ax)$	$\tan ax$	$a\sec^2 ax$
$\frac{1}{a}\ln\left(\tan\frac{ax}{2}\right)$	$\operatorname{cosec} ax$	$-a\operatorname{cosec} ax \cot ax$
$\frac{1}{a}\ln(\sec ax + \tan ax)$	$\sec ax$	$a\sec ax \tan ax$
$\frac{1}{a}\ln(\sin ax)$	$\cot ax$	$-a\operatorname{cosec}^2 ax$
$\frac{1}{a}\cosh ax$	$\sinh ax$	$a\cosh ax$
$\frac{1}{a}\sinh ax$	$\cosh ax$	$a\sinh ax$
$\frac{1}{a}\ln \cosh ax$	$\tanh ax$	$a\,\operatorname{sech}^2 ax$
$\sin^{-1}\left[\frac{x}{a}\right]$	$\frac{1}{\sqrt{a^2-x^2}}$	
$\frac{1}{a}\tan^{-1}\left[\frac{x}{a}\right]$	$\frac{1}{x^2+a^2}$	
$\sinh^{-1}\left[\frac{x}{a}\right]$	$\frac{1}{\sqrt{a^2+x^2}}$	
$\cosh^{-1}\left[\frac{x}{a}\right]$	$\frac{1}{\sqrt{x^2-a^2}}$	