UNIVERSITY OF BOLTON

SCHOOL OF HEALTH & SOCIAL SCIENCES

MATHEMATICS PATHWAY

SEMESTER 1 EXAMINATIONS 2009/10

NUMERICAL ANALYSIS

MODULE NO: MAS2504

Date: Friday 21 January 2010 **Time: 2.00 - 4.15pm**

INSTRUCTIONS TO CANDIDATES:

1. **There are SIX questions.**
2. **Answer FOUR questions.**
3. **All questions carry equal marks.**
4. **Maximum marks for each part/question are shown in brackets.**
5. **Formulae sheets are attached.**

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1. (a) (i) Show that the equation

$$x^2 + 3x - 7 = 0$$

has a real root in the interval [1,2].

(2 marks)

(ii) Show that the following three iterative methods are valid rearrangements of the above equation, and by considering the convergence criteria for each method, using an initial estimate of the root of $x_0 = 1.2$, state which would be the best method to use.

Method 1 $x_{i+1} = \sqrt{7 - 3x_i}$

Method 2 $x_{i+1} = \frac{1}{3}\left(7 - x_i^2\right)$

Method 3 $x_{i+1} = \frac{7}{x_i + 3}$

(11 marks)

(iii) Write down the first six iterates obtained by using this method.

(4 marks)

(b) Show that Aitken's $\triangle^2$ process may be written as

$$X_{\text{imp}} = x_n + \frac{\triangle x_n}{1 - C_L}$$

where C_L is the linear convergence coefficient.

(8 marks)

Please turn the page

2. (a) Show that the rate of convergence of the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for obtaining a root of $f(x) = 0$ is quadratic provided that the root is not multiple.

(12 marks)

(b) The function $f(x) = x^4 - 11x - 5$ has a simple root close to $x = 2$. Use the Newton-Raphson method to find this root correct to four decimal places.

(6 marks)

(c) The modified Newton-Raphson method for multiple roots is

$$x_{n+1} = x_n - \frac{mf(x_n)}{f'(x_n)}$$

where m is the multiplicity of the root.

Show that this method converges directly to the exact triple root of

$$x^3 - 12x^2 + 48x - 64 = 0$$

and write down the value of the triple root.

(7 marks)

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3. (a) Write down the second order Lagrange interpolating polynomial.

(2 marks)

(b) The integral, $I = \int_0^k \sin(x) e^{-x^2} dx$, cannot be evaluated by analytical methods. The following values for $I(k)$ have been calculated to four decimal place accuracy using a numerical method;

$$I(1) = 0.2947$$
$$I(2) = 0.4212$$
$$I(3) = 0.4244$$

Use Lagrange interpolation on this data to obtain an approximation for $I(2.5)$ working to four decimal places.

(10 marks)

(c) Show that:

(i) only one polynomial of degree n or less passes through $n+1$ distinct data points (x_i, y_i), $i = 0, 1, ..., n$;

(7 marks)

(ii) the Lagrange interpolating polynomial $p_n(x)$ passes through the distinct data points (x_i, y_i) $i = 0, 1, ..., n$.

(6 marks)

Please turn the page

4. The Newton-Gregory forward polynomial is given by

$$f(x_s) = \sum_{r=0}^{s} {}^{s}C_r \, \triangle^r f(x_0), \quad s \in \{0, 1, 2, \ldots\}.$$

(a) Use the first three terms of the Newton-Gregory forward polynomial to derive Simpson's rule

$$\int_{x_0}^{x_2} f(x)dx \simeq \frac{h}{3}\{f(x_0) + 4f(x_1) + f(x_2)\}$$

where h is the interval width.

(10 marks)

(b) (i) Obtain the composite form of the above quadrature formula.

(4 marks)

(ii) Use the above to evaluate

$$\int_0^1 e^{-2x^2}\, dx$$

correct to two decimal places.

(11 marks)

Please turn the page

5. The nth degree Chebyshev polynomial can be defined recursively by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

with $T_0(x) = 1$ and $T_1(x) = x$.

(a) Use this definition to find $T_3(x)$, and find the roots of $T_3(x) = 0$.

(5 marks)

(b) By using the substitution $x = \cos z$, show that

$$I = \int_0^{\pi} e^{-2\cos z} dz = \int_{-1}^{1} \frac{e^{-2x}}{\sqrt{1-x^2}} dx.$$

(7 marks)

(c) Use $\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \simeq \frac{\pi}{3} \sum_{i=1}^{3} f(x_i)$, where x_i are the roots of $T_3(x) = 0$ to evaluate $\int_0^{\pi} e^{-2\cos z} dz$.

(7 marks)

(d) Show that if $x_m,\ m = 1, 2, ..., n$ are the roots of $T_n(x) = 0$ then

$$x_m = \cos\left\{ \frac{(2m-1)\pi}{2n} \right\}$$

(6 marks)

Please turn the page

6. (a) Solve the following set of equations by Gauss elimination

$$\begin{aligned} x + 3y + 2z &= 13 \\ 2x - y + 4z &= 12 \\ 3x + 2y - z &= 4 \end{aligned}$$

(12 marks)

(b) (i) The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3 \end{aligned}$$

may be solved using Gauss-Seidel iteration.

Write down the iterative method to find x_i, $i = 1,2,3$.

Show that a sufficient condition for convergence is given by

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{3} |a_{ij}| \quad i = 1, 2, 3.$$

(5 marks)

(ii) Use Gauss-Seidel iteration to solve the following equations correct to two decimal places.

$$\begin{aligned} 3x_1 + 8x_2 - x_3 &= 1 \\ 5x_1 - x_2 - x_3 &= 1 \; . \\ -2x_1 - x_2 + 5x_3 &= 1 \end{aligned}$$

(8 marks)

END OF QUESTIONS

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Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sec^2\theta = 1 + \tan^2\theta$$
$$\mathrm{cosec}^2\,\theta = 1 + \cot^2\theta$$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$

Taylor's Series $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \ldots$$

Maclaurin's Series $$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}f^{(n-1)}(0)}{(n-1)!} + \ldots$$

Root Finding

Aitken's Δ^2-process $$x_{improved} = x_n - \frac{(x_n - x_{n-1})^2}{x_n - 2x_{n-1} + x_{n-2}}$$

Bairstow's Method

$$\begin{aligned} b_1 &= a_1 & c_1 &= b_1 \\ b_2 &= a_2 + rb_1 & c_2 &= b_2 + rc_1 \\ b_{i+1} &= a_{i+1} + rb_i + sb_{i-1} & c_i &= b_i + rc_{i-1} + sc_{i-2} \\ i &= 2, 3, \ldots, n & i &= 3, 4, \ldots, n \end{aligned}$$

$$dr = \frac{b_n c_{n-1} - b_{n+1} c_{n-2}}{c_n c_{n-2} - c_{n-1}^2}$$

$$ds = \frac{b_{n+1} c_{n-1} - b_n c_n}{c_n c_{n-2} - c_{n-1}^2}$$

Interpolation

Lagrange Interpolation $$P_N(x) = \sum_{j=0}^{N} L_j(x) y_j$$

$$L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{x - x_i}{x_j - x_i}$$

Numerical Integration

Trapezium Rule $$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[f(x_0) + f(x_n) + 2\sum_{j=1}^{n-1} f(x_j) \right]$$

Simpson's Rule $$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\left[f(x_0) + f(x_n) + 4\sum_{j=1}^{\frac{N}{2}} f(x_{2j-1}) + 2\sum_{j=1}^{\frac{N}{2}-1} f(x_{2j}) \right]$$

Simpson's Three-Eighths Rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3}{8}h[y_0 + 3y_1 + 3y_2 + 2y_3 + ... + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Calculus

$\int y\,dx$	y	$\dfrac{dy}{dx}$
$\left.\begin{array}{ll}\dfrac{x^{n+1}}{n+1} & n \neq -1 \\ \log_e x & n = -1\end{array}\right\}$	x^n	nx^{n-1}
$-\frac{1}{a}\cos ax$	$\sin ax$	$a\cos ax$
$\frac{1}{a}\sin ax$	$\cos ax$	$-a\sin ax$
$\frac{1}{a}\log_e \sec ax$	$\tan ax$	$a\sec^2 ax$
$\log_e \tan\frac{x}{2}$	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\log_e[\sec x + \tan x]$	$\sec x$	$\sec x \tan x$
$\log_e \sin x$	$\cot x$	$-\operatorname{cosec}^2 x$
$x\log_e x - x$	$\log_e x$	$\frac{1}{x}$
$\frac{1}{a}e^{ax}$	e^{ax}	ae^{ax}
$\dfrac{a^x}{\log_e a}$	$a^x \quad a > 0$	$a^x \log_e x$
$\cosh x$	$\sinh x$	$\cosh x$
$\sinh x$	$\cosh x$	$\sinh x$
$\sin^{-1}x$	$\dfrac{1}{\sqrt{(1-x^2)}}$	
$\tan^{-1}x$	$\dfrac{1}{1+x^2}$	